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THREE THEOREMS ON COMMON SPLITTING FIELDS OF CENTRAL SIMPLE ALGEBRAS

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ABSTRACT

Let $A_1, \dots, A_n$ be central simple algebras over a field F. Suppose that we possess information on the Schur indexes of some tensor products of (some tensor powers of) the algebras. What can be said (in general) about possible degrees of finite field extensions of F splitting the algebras? In Part I, we prove a positive result of that kind. In Part II, we prove a negative result. In Part III, we develop a general approach.

We are working with finite-dimensional central simple algebras over fields and using the standard terminology concerning them. In particular, if A is such an algebra, the **degree** of A is the squareroot of its dimension over the center, the **index** of A is its Schur index (i.e. the minimal possible degree of a field extension of the center splitting A), and the **exponent** of A is the order of its class in the Brauer group of the center.

Part I. A generalization of the Albert–Risman theorems

Let A be a central simple algebra of a prime degree p over a field F and let $B_1, \dots, B_{p-1}$ be central simple F-algebras of degrees $p^{n_1}, \dots, p^{n_{p-1}}$. We show that if every tensor product $A \otimes_F B_i$ has zero divisors, then there exists a field extension E/F of degree $\leq p^{n_1+\cdots+n_{p-1}}$ which splits the algebras $B_1, \dots, B_{p-1}$ as well as the algebra A. In the case $p = 2$, this statement was proved in 1975 by L. Risman ([19]); in the case $p = 2$ and $n_1 = 1$, it is a classical theorem of A. Albert (see [1] or [2]).

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0. Introduction

A well-known theorem of A. Albert states (see [1] or [2]): if the tensor product of two quaternion algebras has zero divisors, then the quaternion algebras can be split by a common extension of the base field of degree ≤ 2.

In 1975, L. J. Risman gave the following generalization of the Albert theorem ([19, Theorem 1]): if the tensor product of a degree 2^n (where $n \geq 1$) central simple algebra A and a quaternion algebra B has zero divisors, then A and B possess a common splitting field of degree $\leq 2^n$.

Attempts to find a generalization of the Risman theorem to the case of an odd prime p were unsuccessful for a long time. Even worse: in 1993 B. Jacob and A. R. Wadsworth ([9], see also Part II) had shown that already the Albert theorem has no generalization to the case of *two* degree p algebras. They found two degree p central simple algebras A, B with no common splitting field of degree p, possessing the following property: for any integers $i, j \geq 0$ the index of the tensor product $A^{\otimes i} \otimes B^{\otimes j}$ was $\leq p$.

We propose here the following generalization of the Risman theorem:

THEOREM I.1: *Let A be a central simple algebra of a prime degree p over a field F and let $B_1, \ldots, B_{p-1}$ be central simple F-algebras of degrees $p^{n_1}, \ldots, p^{n_{p-1}}$. Set $n \stackrel{\text{def}}{=} n_1 + \cdots + n_{p-1}$ and suppose that for every $i = 1, \ldots, p-1$ the tensor product $A \otimes_F B_i$ has zero divisors. Then there exists a field extension E/F of degree $\leq p^n$ which splits all the algebras $A, B_1, \ldots, B_{p-1}$.*

In the particular case where $n_1 = \cdots = n_{p-1} = 1$, Theorem I.1 can be regarded as a generalization of the Albert theorem. For instance, taking $p = 3$ we get the following

Example I.2: Let A, B, C be three degree 3 central simple algebras over a field F. Suppose that each of the two tensor products $A \otimes B$ and $A \otimes C$ has zero divisors. Then there exists a field extension E/F of degree ≤ 9 which splits all the three algebras A, B, C.

In fact, a general method of obtaining results of the type similar to Theorem I.1 is developed in Part III. However, this method allows one to control degrees of common splitting fields of algebras only up to a prime to p factor. In order to obtain the announced exact statement, we use here a refinement of that method. It is based on the intersection theory and especially on the theory of *non-negative intersections* developed in [5, Chapter 12] (see the proof of Proposition I.4).

Terminology and notation: Let F be a field. We fix an algebraic closure $\bar{F}$ of F and, for any F-variety X, write $\bar{X}$ for the $\bar{F}$-variety $X_{\bar{F}}$.

Let σ be a cycle on X. We write $[\sigma]$ for the class of σ in the Chow group $\mathrm{CH}^*(X)$ and $\bar{\sigma}$ for the corresponding cycle on $\bar{X}$. Sometimes, while working on $X \times Y$, where Y is another F-variety, abusing notation, we denote by σ as well the cycle $\sigma \times Y$.

The **degree** $\deg(\sigma)$ of a simple 0-dimensional cycle σ is the degree of its residue field over the base field. The degree of an arbitrary 0-dimensional cycle $\sigma = \sum l_j \sigma_j$, where l_j are integers and σ_j are simple cycles, is defined as $\sum l_j \deg(\sigma_j)$.

A cycle $\sigma = \sum l_j \sigma_j$ (of any dimension) is called **non-negative**, if all the integers l_j are non-negative.

1. Preliminaries

In this section we prove two (independent) statements needed for the next section.

LEMMA I.3: *For any integers $n, m \geq 0$, let $\phi : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ be the Segre imbedding. Denote by $f \in \mathrm{CH}^1(\mathbb{P}^n)$, $g \in \mathrm{CH}^1(\mathbb{P}^m)$, and $h \in \mathrm{CH}^1(\mathbb{P}^{nm+n+m})$ the classes of hyperplanes. Then $\phi^*(h) = f + g \in \mathrm{CH}^1(\mathbb{P}^n \times \mathbb{P}^m)$, where $\phi^* : \mathrm{CH}^*(\mathbb{P}^{nm+n+m}) \to \mathrm{CH}^*(\mathbb{P}^n \times \mathbb{P}^m)$ is the pull-back homomorphism.*

Proof: Denote by $[x_0 : \cdots : x_n]$, $[y_0 : \cdots : y_m]$, and $[z_0 : \cdots : z_{nm+n+m}]$ the coordinates in $\mathbb{P}^n$, $\mathbb{P}^m$, and $\mathbb{P}^{nm+n+m}$. The Segre embedding ϕ is determined by the rule

$$\phi([x_0 : \cdots : x_n] \times [y_0 : \cdots : y_m]) = [x_0 y_0 : x_0 y_1 : \cdots : x_n y_{m-1} : x_n y_m].$$

The intersection of the hyperplane $z_0 = 0$ with $\mathbb{P}^n \times \mathbb{P}^m$ has two transversal components: one of them is determined in $\mathbb{P}^n \times \mathbb{P}^m$ by the equation $x_0 = 0$ and represents f while the other one is determined by the equation $y_0 = 0$ and represents g. ∎

PROPOSITION I.4: *Let T be a direct product of Severi–Brauer varieties over a field F. Let σ and σ' be non-negative cycles on T and let $\psi : T' \to T$ be a closed imbedding. Then*

(1) *there exists a non-negative cycle τ on T such that $[\tau] = [\sigma] \cdot [\sigma'] \in \mathrm{CH}^*(T)$;*
(2) *there exists a non-negative cycle τ' on T' such that $\psi^*([\sigma]) = [\tau']$.*

Proof: According to [5, Corollary 12.2a], both the assertions hold if the tangent bundle of T is generated by its global sections. In order to check this condition on the tangent bundle for a variety over F, it suffices to check it over an extension of F, e.g., over $\bar{F}$. Since the $\bar{F}$-variety $\bar{T}$ is isomorphic to a direct product of projective spaces, the tangent bundle of $\bar{T}$ is generated by its global sections (see [5, Examples 12.2.1a and 12.2.1c]).

2. The proof

In this section we prove Theorem I.1.

We denote by X, $Y_1, \dots, Y_{p-1}$ and $T_1, \dots, T_{p-1}$ the Severi–Brauer varieties of the algebras A, $B_1, \dots, B_{p-1}$ and $A \otimes_F B_1, \dots, A \otimes_F B_{p-1}$. We set

$$Y \stackrel{\text{def}}{=} \prod_{i=1}^{p-1} Y_i \,.$$

For every i, the tensor product of ideals induces a closed imbedding ψ_i: $X \times Y_i \hookrightarrow T_i$ which is a twisted form of the Segre imbedding $\mathbb{P}^{p-1} \times \mathbb{P}^{p^{n_i}-1} \hookrightarrow \mathbb{P}^{p^{n_i+1}-1}$.

Let $f \in \mathrm{CH}^1(\bar{X})$, $g_i \in \mathrm{CH}^1(\bar{Y}_i)$, and $h_i \in \mathrm{CH}^1(\bar{T}_i)$ be the classes of hyperplanes.

The algebra $A \otimes B_i$ (for every i) has degree p^{n_i+1} and zero divisors, so that its index divides p^{n_i}. Therefore, there exists a simple (and in particular non-negative) p^{n_i}-codimensional cycle σ_i on the variety T_i such that $[\bar{\sigma}_i] = h_i^{p^{n_i}} \in \mathrm{CH}^{p^{n_i}}(\bar{T}_i)$ (see [3, §3]).

By item (2) of Proposition I.4, there exists a non-negative cycle τ_i on $X \times Y_i$ such that $[\tau_i] = \psi_i^*([\sigma_i])$. Since $[\bar{\sigma}_i] = h_i^{p^{n_i}}$, it follows from Lemma I.3 that $[\bar{\tau}_i] = (f + g_i)^{p^{n_i}} \in \mathrm{CH}^{p^{n_i}}(\bar{X} \times \bar{Y}_i)$.

Applying item (1) of Proposition I.4 to the variety $X \times Y$, we find a non-negative cycle τ on $X \times Y$ such that $[\tau] = [\tau_1] \cdots [\tau_{p-1}] \in \mathrm{CH}^*(X \times Y)$. Note that τ is a cycle of codimension $p^{n_1} + \cdots + p^{n_{p-1}}$ on a variety of dimension $(p-1) + (p^{n_1} - 1) + \cdots + (p^{n_{p-1}} - 1) = p^{n_1} + \cdots + p^{n_{p-1}}$, so that it is a 0-dimensional cycle. Moreover,

$$\begin{aligned}[\bar{\tau}] = [\bar{\tau}_1] \cdots [\bar{\tau}_{p-1}] &= (f + g_1)^{p^{n_1}} \cdots (f + g_{p-1})^{p^{n_{p-1}}} \\ &= p^n \cdot (f^{p-1} g_1^{p^{n_1}-1} \cdots g_{p-1}^{p^{n_{p-1}}-1}),\end{aligned}$$

where the last equality holds because of the relations $f^p = 0$ and $g_i^{p^{n_i}} = 0$ (for $i = 1, \dots, p-1$). Since f^{p-1} is the class of a rational point on $\bar{X}$ and since $g_i^{p^{n_i}-1}$ is the class of a rational point on the variety $\bar{Y}_i$ for every i, we get $\deg(\tau) = p^n$.

Let τ' be any simple cycle included in τ. Since the cycle τ is non-negative, we have $\deg(\tau') \le p^n$. The residue field E of τ' is a common splitting field of the algebras $A, B_1, \dots, B_{p-1}$ and $[E : F] = \deg(\tau') \le p^n$. The proof of Theorem I.1 is complete.

Part II. Linked algebras

Two central simple algebras A, B of a prime degree p over a field are called **linked**, if for any integers $i, j \geq 0$ the index of $A^{\otimes i} \otimes B^{\otimes j}$ is at most p. They are called **strongly linked**, if they possess a common splitting field of a finite degree (over the base field) not divisible by p^2.

We show that for any two not strongly linked central simple algebras, an extension of the base field can be made such that the algebras become **linked** while still being not **strongly linked**.

The initial idea to attack the problem using the 0-dimensional Chow group of the product of Severi–Brauer varieties was proposed to the author by O. Izhboldin.

0. Introduction

As already mentioned in Part I, in 1931 A. Albert proved (see [1] or [2]): two quaternion division algebras can be split by a common quadratic extension of the base field provided that their tensor product has zero divisors.

Attempts to generalize the theorem of Albert to the case of an odd prime p led in 1987 to counter-examples of J.-P. Tignol and A. R. Wadsworth ([22, Proposition 5.1]), who constructed two degree p central division algebras A, B with zero divisors in $A \otimes B$ and without common splitting field extensions of degree p.

Stronger counter-examples was obtained in 1993 by B. Jacob and A. R. Wadsworth ([9]). They found two degree p central division algebras A, B without common splitting field of degree p possessing the following property: for any integers $i, j \geq 0$ the index of the tensor product $A^{\otimes i} \otimes B^{\otimes j}$ was $\leq p$. It was in fact even shown that any common splitting field of the algebras A, B has degree divisible by p^2 ([9, Remark 2]).

In this Part we show (see Theorem II.1) that similar counter-examples can be obtained by an appropriate base extension starting from *any* two degree p central simple algebras A, B provided that the degree of every common splitting field of A, B is divisible by p^2 (which is guaranteed, e.g., if the tensor product $A \otimes B$ is a division algebra). The proof is essentially different from that of [9].

Notation: Let F be a field and let X be a smooth variety over F. We write $K(X)$ for the Grothendieck group of X; $K(X)^{(n)}$, where $n \geq 0$, for the n-codimensional term of the topological filtration on X (see [18, §7] for a definition of the topological filtration, which is called the **filtration by codimension of support** in the reference); $\mathrm{CH}^*(X)$ for the Chow ring of X graded by codimension of cycles; $\mathrm{CH}_0(X)$ for the 0-dimensional component of the Chow ring, i.e. for the component $\mathrm{CH}^{\dim X}(X)$.

Let A be a central simple F-algebra. We write $\mathrm{SB}(A)$ for the Severi–Brauer variety of A and $\mathrm{SB}(r, A)$ (with $r \geq 0$) for the generalized Severi–Brauer varieties (also called **generic partial splitting varieties**) of A.

1. The theorem

Throughout this Part, A, B are central simple algebras of a prime degree p over a field F. We call them **linked**, if $\mathrm{ind}(A^{\otimes i} \otimes_F A^{\otimes j}) \leq p$ for any $i, j \geq 0$. The algebras A and B are called **strongly linked**, if there exists a finite field extension E/F such that

- the algebras A_E and B_E are split, and
- the degree $[E : F]$ is not divisible by p^2.

Clearly, strongly linked algebras are linked. By the theorem of Albert, the inverse is true for $p = 2$. For any odd p, we shall prove the following

THEOREM II.1: *Let F be a field, p be an odd prime number, and A, B be degree p central simple F-algebras. Suppose that A, B are not strongly linked. Then there exists a field extension $\tilde{F}/F$ such that the algebras $A_{\tilde{F}}, B_{\tilde{F}}$ are linked but still not strongly linked.*

One can take as $\tilde{F}$ the function field $F(T)$ of the following product of generalized Severi–Brauer varieties:

$$T \stackrel{\text{def}}{=} \mathrm{SB}(p, A \otimes B) \times \mathrm{SB}(p, A^{\otimes 2} \otimes B) \times \cdots \times \mathrm{SB}(p, A^{\otimes p-1} \otimes B).$$

2. Preliminary observations

We set $X \stackrel{\text{def}}{=} \mathrm{SB}(A) \times \mathrm{SB}(B)$. Let L/F be an arbitrary common splitting field extension of A, B.

LEMMA II.2: *The algebras A, B are not strongly linked if and only if the image of the restriction homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L)$ is divisible by p^2.*

Proof: Note that since the algebras A_L, B_L are split, the variety X_L is isomorphic to a product of two projective spaces. Therefore the degree map $\deg : \mathrm{CH}_0(X_L) \to \mathbb{Z}$ is an isomorphism. Also note that the composition $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L) \to \mathbb{Z}$ of the restriction and degree maps is the degree map for $\mathrm{CH}_0(X)$.

If the image of the restriction homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L)$ is not divisible by p^2, then there exists a closed point on X of degree not divisible by p^2. The residue field of this point is a common splitting field for A and B showing that the algebras are strongly linked.

To prove the inverse implication, suppose the image of the restriction homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L)$ is divisible by p^2. Then the image of the degree homomorphism $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is divisible by p^2 as well. Let E/F be any finite field extension such that the algebras A_E, B_E are split. The variety X_E has then a rational point. Taking the transfer, we obtain a 0-dimensional cycle of degree $[E:F]$ on X. Consequently, $[E:F]$ is divisible by p^2, i.e. the algebras A, B are not strongly linked. ∎

LEMMA II.3: *If the algebras A, B are not strongly linked, then, for any not simultaneously 0 integers $0 \le i, j < p$, the index of the tensor product $A^{\otimes i} \otimes B^{\otimes j}$ is divisible by p.*

Proof: Let i, j be any integers such that $0 \le i, j < p$. Suppose that the tensor product $A^{\otimes i} \otimes B^{\otimes j}$ is split. If $i \ne 0$, then any splitting field of B splits A as well; therefore the algebras A, B possess a common splitting field of degree p in this case, which contradicts the assumption they are not strongly linked. Thus $i = 0$. The symmetric argument shows that $j = 0$ as well. ∎

LEMMA II.4: *Suppose that, for any not simultaneously 0 integers $0 \le i, j < p$, the index of tensor product $A^{\otimes i} \otimes B^{\otimes j}$ is divisible by p. Then the image of the restriction homomorphism $K(X)^{(1)} \to K(X_L)^{(1)}$ is divisible by p.*

Proof: We have already noticed that since the algebras A_L and B_L are split, the varieties $\mathrm{SB}(A_L)$ and $\mathrm{SB}(B_L)$ are isomorphic to ($(p-1)$-dimensional) projective spaces. Let ξ, η be the Grothendieck classes of the tautological line bundles on $\mathrm{SB}(A_L), \mathrm{SB}(B)_L$ respectively. The group $K(X_L)$ is generated by $\{\xi^i\eta^j\}_{i,j=0}^{p-1}$. Using the generalized version [17, Proposition 3.1] of Quillen's computation of K-theory for Severi–Brauer schemes [18, Theorem 4.1 of §8], one can show that the image of the restriction map $K(X) \to K(X_L)$ is generated by

$$\{\mathrm{ind}(A^{\otimes i} \otimes B^{\otimes j}) \cdot \xi^i\eta^j\}_{i,j=0}^{p-1} .$$

Since the first term of the topological filtration coincides with the kernel of the rank map, the group $K(X_L)^{(1)}$ is generated by $\{\xi^i\eta^j - 1\}_{i,j=0}^{p-1}$ while the image of $K(X)^{(1)} \to K(X_L)^{(1)}$ is generated by $\{\mathrm{ind}(A^{\otimes i} \otimes B^{\otimes j}) \cdot (\xi^i\eta^j - 1)\}_{i,j=0}^{p-1}$.

The assertion required follows now from the assumption on the indexes $\mathrm{ind}(A^{\otimes i} \otimes B^{\otimes j})$. ∎

COROLLARY II.5: *In the conditions of Lemma II.4, for any $n > 0$, the image of the restriction homomorphism $\mathrm{CH}^n(X) \to \mathrm{CH}^n(X_L)$ is divisible by p.*

Proof: For any $n \geq 0$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^n(X_L) & \longrightarrow & K(X_L)^{(n)}/K(X_L)^{(n+1)} \\ {\scriptstyle \mathrm{res}_{L/F}}\uparrow & & \uparrow{\scriptstyle \mathrm{res}_{L/F}} \\ \mathrm{CH}^n(X) & \longrightarrow & K(X)^{(n)}/K(X)^{(n+1)} \end{array}$$

where the horizontal arrows are the canonical epimorphisms (see [21, §9] for their definition and basic properties). Since X_L is a direct product of projective spaces, the groups $\mathrm{CH}^n(X_L)$ are torsion-free (see, for example, [7, §2 of Chapter A]), whereas the upper arrow is an isomorphism. Therefore, it suffices to show that for any $n > 0$ the image of the restriction homomorphism $K(X)^{(n)} \to K(X_L)^{(n)}$ is divisible by p. By Lemma II.4, the image of $K(X)^{(n)} \to K(X_L)^{(1)}$ is divisible by p. Since the quotient $K(X_L)^{(1/n)}$ is torsion-free, we are done. ∎

3. The proof

In this section we prove Theorem II.1.

We set $\tilde{F} \stackrel{\mathrm{def}}{=} F(T)$, $\tilde{A} \stackrel{\mathrm{def}}{=} A_{\tilde{F}}$, and $\tilde{B} \stackrel{\mathrm{def}}{=} B_{\tilde{F}}$, where T is the product of generalized Severi–Brauer varieties written down at the end of Section 1. We also set $X \stackrel{\mathrm{def}}{=} \mathrm{SB}(A) \times \mathrm{SB}(B)$ and $\tilde{X} \stackrel{\mathrm{def}}{=} \mathrm{SB}(\tilde{A}) \times \mathrm{SB}(\tilde{B})$.

First of all we note that by the main property of generalized Severi–Brauer varieties one has $\mathrm{ind}(\tilde{A}^{\otimes i} \otimes \tilde{B}) \leq p$ for any $i = 1, \ldots, p-1$. Therefore $\mathrm{ind}(\tilde{A}^{\otimes i} \otimes \tilde{B}^{\otimes j}) \leq p$ for any integers $i, j \geq 0$, i.e. the algebras $\tilde{A}, \tilde{B}$ are linked. So we only have to show that they are not strongly linked.

LEMMA II.6: *For any not simultaneously 0 integers $0 \leq i, j < p$, the index of tensor product $\tilde{A}^{\otimes i} \otimes \tilde{B}^{\otimes j}$ is divisible by p.*

Proof: First of all, by Lemma II.3, the assertion holds for the algebras A, B (instead of $\tilde{A}, \tilde{B}$), because they are assumed to be not strongly linked. Applying the computation [4, Theorem 7]* of the relative Brauer group of the function

* See also [15, Corollary 2.7] where this computation is done in a more general context; it is also an easy consequence of the index reduction formula [20, Theorem 2.13] (see the footnote in the proof of Lemma III.2).

field of a generalized Severi–Brauer variety, one sees that the relative Brauer group $\ker\big(\mathrm{Br}(F) \to \mathrm{Br}(F(T))\big)$ of the function field of the variety T is trivial. Therefore the assertion on $\tilde{A}, \tilde{B}$ holds as well. ∎

Let L/F be a common splitting field extension for the algebras A, B. Set $\tilde{L} \stackrel{\mathrm{def}}{=} L(T)$. Clearly, $\tilde{L}/\tilde{F}$ is a common splitting field extension for $\tilde{A}, \tilde{B}$.

Corollary II.7: *For any $n > 0$, the image of the restriction homomorphism $\mathrm{CH}^n(\tilde{X}) \to \mathrm{CH}^n(\tilde{X}_{\tilde{L}})$ is divisible by p.*

Proof: Follows from Lemma II.6 and Corollary II.5. ∎

We consider the graded ring $\mathrm{CH}^*(\tilde{X})$ as a graded $\mathrm{CH}^*(X)$-algebra via the restriction homomorphism $\mathrm{CH}^*(X) \to \mathrm{CH}^*(\tilde{X})$.

Proposition II.8: *The $\mathrm{CH}^*(X)$-algebra $\mathrm{CH}^*(\tilde{X})$ is generated by its graded components of codimensions $\leq p$.*

Proof: Since the pull-back $\mathrm{CH}^*(X \times T) \to \mathrm{CH}^*(\tilde{X})$ is an epimorphism of graded $\mathrm{CH}^*(X)$-algebras (see [12, Theorem 3.1] or [8, Proposition 5.1]), it suffices to show that the algebra $\mathrm{CH}^*(X \times T)$ is generated by its graded components of codimensions $\leq p$.

Consider the variety $X \times T$ as a scheme over X via the projection. According to [11, Proposition 5.3], it is a product (over X) of p-grassmanians. By [5, Proposition 14.6.5], $\mathrm{CH}^*(X \times T)$ is therefore generated (as a $\mathrm{CH}^*(X)$-algebra) by the Chern classes of the tautological bundles on the grassmanians. Since all these bundles have rank p, they may have non-trivial Chern classes only in codimensions $\leq p$. ∎

Finally, Theorem II.1 follows from Lemma II.2 and the following assertion:

Corollary II.9: *The image of the restriction $\mathrm{CH}_0(\tilde{X}) \to \mathrm{CH}_0(\tilde{X}_{\tilde{L}})$ is divisible by p^2.*

Proof: Note that since $p \neq 2$, we have $p < (p-1)^2 = \dim X$. Thus, by Proposition II.8, the group $\mathrm{CH}_0(\tilde{X})$ is generated by the image of $\mathrm{CH}_0(X) \to \mathrm{CH}_0(\tilde{X})$ and by the products $\mathrm{CH}^n(\tilde{X}) \cdot \mathrm{CH}^m(\tilde{X})$ with $n, m > 0$.

The image in $\mathrm{CH}_0(\tilde{X}_{\tilde{L}})$ of the first part of the generators is divisible by p^2, since in the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0(\tilde{X}) & \longrightarrow & \mathrm{CH}_0(\tilde{X}_{\tilde{L}}) \\ \uparrow & & \uparrow \\ \mathrm{CH}_0(X) & \longrightarrow & \mathrm{CH}_0(X_L) \end{array}$$

the image of the bottom arrow is divisible by p^2 (Lemma II.2).

The image in $\mathrm{CH}_0(\tilde{X}_{\tilde{L}})$ of the second part of the generators is divisible by p^2, since for any $n > 0$ the image of $\mathrm{CH}^n(\tilde{X}) \to \mathrm{CH}^n(\tilde{X}_{\tilde{L}})$ is divisible by p (Corollary II.7). ∎

Part III. A general approach

For every prime number p and every map α: $\mathbb{Z}^n \to \mathbb{Z}$, we find the minimal integer β such that the following assertion holds: any elements $x_1, \dots, x_n$ of the Brauer group $\mathrm{Br}(F)$ of an arbitrary field F, satisfying the conditions $\mathrm{ind}(i_1 x_1 + \cdots + i_n x_n) = p^{\alpha(i_1,\dots,i_n)}$ for all $i_1, \dots, i_n \in \mathbb{Z}$, possess a finite common splitting field extension E/F with $v_p([E:F]) \leq \beta$, where v_p denotes the multiplicity of p.

0. Introduction

Let us fix a prime number p. Let α: $\mathbb{Z}^n \to \mathbb{Z}$ be an arbitrary map. We say that α is the **behaviour** of elements $x_1, \dots, x_n$ of the Brauer group $\mathrm{Br}(F)$ of a field F, if for any $i_1, \dots, i_n \in \mathbb{Z}$ the Schur index $\mathrm{ind}(i_1 x_1 + \cdots + i_n x_n)$ equals $p^{\alpha(i_1,\dots,i_n)}$. We say that α is a **behaviour**, if there exists a field F and elements $x_1, \dots, x_n \in \mathrm{Br}(F)$ with the behaviour α.

Let F be a field and $x \in \mathrm{Br}(F)$. A **splitting field** of x is by definition a field extension E of F such that $x_E = 0 \in \mathrm{Br}(E)$. A **common splitting field** of several elements of $\mathrm{Br}(F)$ is by definition a field which is a splitting field for each of the elements. We consider only (common) splitting fields **finite** over the base field.

Let us fix a behaviour α. In this note we determine the minimal integer β such that the following assertion holds (see Theorem III.5): if some elements in the Brauer group of an arbitrary field F have the behaviour α, then they possess a common splitting field E with $v_p([E:F]) \leq \beta$.

Similar questions were considered earlier in the literature. Here is a list of known results:

1. A classical theorem of Albert (see [1] or [2]) states: if the tensor product of two quaternion division algebras has zero divisors, then the quaternion algebras possess a common splitting field quadratic over the base field.
2. A generalization of the Albert theorem due to Risman ([19, Theorem 1]) asserts: if the tensor product of a 2-primary division algebra A and a quaternion algebra B has zero divisors, then A and B possess a common splitting field of degree $\deg A$.

3. Jacob and Wadsworth ([9], see also Part II) constructed two division algebras of prime degree p over certain field F such that

- $\operatorname{ind}(A^{\otimes i} \otimes_F B^{\otimes j}) \leq p$ for any $i, j \geq 0$, and
- the degree of any common splitting field of A and B is divisible by p^2.

4. The following was noticed by M. Rost (unpublished). Three quaternion algebras, such that the Brauer class of any tensor product of some of them is represented by a quaternion algebra as well, cannot be (in general) split by a common quadratic extension of the base field.

5. A generalization of the Risman theorem to the case of an odd prime is obtained in Part I.

Note that the theorem presented here does not assert existence of a common splitting field of degree p^β. We do not even know whether such an assertion is true in general. However, in certain particular cases (this means, for certain concrete behaviours) the proof can be refined in order to get the stronger result. An example is the theorem of Part I.

Notation: Let X be a smooth variety over a field F. As in Part II, we write $K(X)$ for the Grothendieck group of X. Besides, we write $\Gamma_0 K(X)$ for the 0-dimensional term of the gamma-filtration, on $K(X)$ (for a definition of the gamma-filtration, see [13, Definition 8.3] or [10, Definition 2.6]); $\mathrm{T}_0\, K(X)$ for the 0-dimensional term of the topological filtration on X. We fix an algebraic closure $\bar{F}$ of F and write $\bar{X}$ for the $\bar{F}$-variety $X_{\bar{F}}$.

For any *projective homogeneous* variety X, we identify $K(X)$ with a subgroup in $K(\bar{X})$ via the restriction homomorphism $K(X) \to K(\bar{X})$ which is injective by [16].

The order of a finite set S is denoted by $|S|$.

As in Part II, we write $\mathrm{SB}(A)$ for the Severi–Brauer variety of a central simple F-algebra A and $\mathrm{SB}(r, A)$ ($r \geq 0$) for the generalized Severi–Brauer varieties of A.

1. "Generic" algebras of given behaviour

For any central simple algebras $A_1, \dots, A_n$ over a field F, we define their behaviour to be the behaviour of their classes in the Brauer group of F.

As in [11, Définition 3.5], we say that algebras $A_1, \dots, A_n$ are **disjoint** if, for any integers $i_1, \dots, i_n \geq 0$,

$$\operatorname{ind}(A^{\otimes i_1} \otimes_F \cdots \otimes_F A^{\otimes i_n}) = \operatorname{ind}(A^{\otimes i_1}) \cdots \operatorname{ind}(A^{\otimes i_n}).$$

We say that a collection of algebras $\tilde{A}_1, \dots, \tilde{A}_n$ is **"generic"** (compare to [11, Définition 5.4]), if it can be obtained via the following procedure. We start with some disjoint central simple algebras $A_1, \dots, A_n$ over a field F such that $\operatorname{ind} A_j = \exp A_j$ for every $j = 1, \dots, n$. Then we take some central simple algebras $B_1, \dots, B_m$ whose Brauer classes lie in the subgroup of $\operatorname{Br}(F)$ generated by the Brauer classes of $A_1, \dots, A_n$. We denote by Y the direct product $\operatorname{SB}(r_1, A_1) \times \cdots \times \operatorname{SB}(r_m, A_m)$ of generalized Severi–Brauer varieties with some $r_1, \dots, r_m \geq 0$ and we set $\tilde{A}_i \overset{\text{def}}{=} (A_i)_{F(Y)}$ for each $i = 1, \dots, n$.

PROPOSITION III.1: *For any behaviour $\alpha\colon \mathbb{Z}^n \to \mathbb{Z}$, there exist "generic" division algebras $\tilde{A}_1, \dots, \tilde{A}_n$ (over a suitable field $\tilde{F}$) having the behaviour α.*

We prove the proposition after the following

LEMMA III.2: *Let A, B be central simple algebras over a field F and let A', B' be central simple algebras over a field F'. Suppose that $\deg B = \deg B'$ and that, for any $i \geq 0$, the index of $A' \otimes_{F'} B'^{\otimes i}$ divides the index of $A \otimes_F B^{\otimes i}$. Then, for any $r \geq 0$, the index of $A'_{F'(\operatorname{SB}(r,B'))}$ divides the index of $A_{F(\operatorname{SB}(r,B))}$.*

Proof: Set $s \overset{\text{def}}{=} \deg B = \deg B'$ and denote by d the greatest common divisor of r and s. By the index reduction formula for the function fields of generalized Severi–Brauer varieties [20, Theorem 2.13],* one has

$$\operatorname{ind}(A_{F(\operatorname{SB}(r,B))}) = \gcd_{1 \leq i \leq s} \left(\frac{d}{\gcd(i,d)} \operatorname{ind}(A \otimes B^{\otimes i}) \right).$$

Replacing A by A' and B by B', we get a formula for $\operatorname{ind}(A'_{F'(\operatorname{SB}(r,B'))})$. Since $\operatorname{ind}(A' \otimes B'^{\otimes i})$ divides $\operatorname{ind}(A \otimes B^{\otimes i})$ for any i, we are done. ∎

Proof of Proposition III.1: We start with disjoint division algebras $A_1, \dots, A_n$ over a suitable field F such that for any $j = 1, \dots, n$ one has

$$\deg A_j = \exp A_j = p^{\alpha(0,\dots,0,1,0,\dots,0)},$$

where 1 (in the argument of α) is placed on the j-th position (algebras like that do definitely exist). For every $i_1, \dots, i_n$ with $0 \leq i_j < \deg A_j$, we consider the algebra

$$B_{i_1 \cdots i_n} \overset{\text{def}}{=} A_1^{\otimes i_1} \otimes \cdots \otimes A_n^{\otimes i_n}$$

* An index reduction formula for the generalized Severi–Brauer varieties was first established by A. Blanchet in [4]. The simpler formula which is used here was found by D. Saltman and proved in a different way in [20]. A. Wadsworth deduced this simple formula from Blanchet's formula in [23]. We refer also to [14], where the simple formula is reestablished in a more general context.

and denote by $Y_{i_1\cdots i_n}$ the variety $\mathrm{SB}(p^{\alpha(i_1,\dots,i_n)}, B_{i_1\cdots i_n})$. We set $Y \stackrel{\mathrm{def}}{=} \prod Y_{i_1\cdots i_n}$ and $\tilde{A}_j \stackrel{\mathrm{def}}{=} (A_j)_{F(Y)}$ for all $j = 1,\dots,n$. We state $\tilde{A}_1,\dots,\tilde{A}_n$ are the "generic" division algebras required.

To show that $\mathrm{ind}(i_1[\tilde{A}_1] + \cdots + i_n[\tilde{A}_n]) = p^{\alpha(i_1,\dots,i_n)}$ for all $i_1,\dots,i_n \in \mathbb{Z}$, it suffices to check that

$$\mathrm{ind}(\tilde{A}_1^{\otimes i_1} \otimes \cdots \otimes \tilde{A}_n^{\otimes i_n}) = p^{\alpha(i_1,\dots,i_n)}$$

for any $i_1,\dots,i_n$ with $0 \le i_j < \deg A_j$. Since the inequality $\le$ is evident, it suffices to prove the inverse inequality.

Since α is a behaviour, we can find division algebras $A'_1,\dots,A'_n$ over a field F' having the behaviour α. Clearly, for any $i_1,\dots,i_n \ge 0$, the index of the algebra $B'_{i_1\dots i_n} \stackrel{\mathrm{def}}{=} A'^{\otimes i_1}_1 \otimes \dots \otimes A'^{\otimes i_n}_n$ equals $p^{\alpha(i_1,\dots,i_n)}$ and divides the index of the algebra $B_{i_1\dots i_n}$ (while their degrees coincide). Let us choose some integers $i'_1,\dots,i'_n$ with $0 \le i'_j \le \deg A'_j$. By Lemma III.2, $\mathrm{ind}(B'_{i_1\cdots i_n})_{F'(Y'_{i'_1\cdots i'_n})}$ divides $\mathrm{ind}(B_{i_1\cdots i_n})_{F(Y_{i'_1\cdots i'_n})}$, where $Y'_{i'_1\cdots i'_n} \stackrel{\mathrm{def}}{=} \mathrm{SB}(p^{\alpha(i'_1,\dots,i'_n)}, B'_{i'_1\cdots i'_n})$. Moreover, the extension $F'(Y'_{i'_1\cdots i'_n})/F$ does not in fact affect the index of any F'-algebra, because the variety $Y'_{i'_1\cdots i'_n}$ is rational. Therefore, the index of $B'_{i_1\cdots i_n}$ itself divides $\mathrm{ind}(B_{i_1\cdots i_n})_{F(Y_{i'_1\cdots i'_n})}$.

So we see that if we replace the base field F of the algebras $A_1,\dots,A_n$ by the function field $F(Y_{i'_1\cdots i'_n})$, the index of every $B_{i_1\cdots i_n}$ is still divisible by $p^{\alpha(i_1,\dots,i_n)}$.

After that we pass to the function field of another variety $Y_{i''_1\cdots i''_n}$. Proceeding in this way, we get in the end the required statement on the indexes. ∎

2. Definition of β

We fix a prime number p and a behaviour $\alpha\colon \mathbb{Z}^n \to \mathbb{Z}$.

Let us consider a field F and elements $x_1,\dots,x_n \in \mathrm{Br}(F)$ with the behaviour α. Choose division F-algebras representing the elements $x_1,\dots,x_n$ and denote by $X_1,\dots,X_n$ the corresponding Severi–Brauer varieties. Set $X \stackrel{\mathrm{def}}{=} X_1 \times \cdots \times X_n$.

Since $\bar{X}$ is isomorphic to a direct product of projective spaces, $\Gamma_0 K(\bar{X})$ is an infinite cyclic group generated by the class of a rational point. We have $0 \ne \Gamma_0 K(X) \subset \Gamma_0 K(\bar{X}) \simeq \mathbb{Z}$. Therefore, the quotient $\Gamma_0 K(\bar{X})/\Gamma_0 K(X)$ is a finite group.

Definition III.3: We put $\beta \stackrel{\mathrm{def}}{=} v_p(|\Gamma_0 K(\bar{X})/\Gamma_0 K(X)|)$.

LEMMA III.4: *The integer β, defined in Definition III.3, depends only on the prime p and on the behaviour α; it does not depend on the choice of the field F and the elements $x_1, \ldots, x_n \in \mathrm{Br}(F)$.*

Proof: According to [11, Corollaire 2.2], the groups $\Gamma_0 K(X)$ and $\Gamma_0 K(\bar{X})$ depend only on p and on the behaviour of $x_1, \ldots, x_n$. ∎

3. The theorem

THEOREM III.5: *For a prime number p and a behaviour α: $\mathbb{Z}^n \to \mathbb{Z}$, let β be the integer defined in the previous section. Then*

(1) *for any field F, any n elements $x_1, \ldots, x_n \in \mathrm{Br}(F)$ with the behaviour α possess a common splitting field E/F satisfying the condition*

$$v_p([E:F]) \leq \beta;$$

(2) *there exists a field F and elements $x_1, \ldots, x_n \in \mathrm{Br}(F)$ with the behaviour α such that all their common splitting fields E/F satisfy the condition*

$$v_p([E:F]) \geq \beta.$$

We prove the theorem after the following

LEMMA III.6: *Let $A_1, \ldots, A_n$ be central simple algebras over a field F and let $X_1, \ldots, X_n$ be the corresponding Severi–Brauer varieties. Set $X \stackrel{\text{def}}{=} X_1 \times \cdots \times X_n$ and $\beta' \stackrel{\text{def}}{=} v_p(|\mathrm{T}_0 K(\bar{X})/\mathrm{T}_0 K(X)|)$. Then:*

(1) *for any common splitting field E/F of $A_1, \ldots, A_n$,*

$$v_p([E:F]) \geq \beta';$$

(2) *the algebras $A_1, \ldots, A_n$ possess a common splitting field E/F with*

$$v_p([E:F]) = \beta'.$$

Proof: For any variety Y, $\mathrm{T}_0 K(Y)$ is by definition the subgroup of $K(Y)$ generated by the classes $[y] \in K(Y)$ of the closed points $y \in Y$. If Y is a complete F-variety, the rule $[y] \mapsto \deg(y) \stackrel{\text{def}}{=} [F(y):F]$, where $F(y)$ is the residue field of y, determines a well-defined homomorphism $\deg\colon \mathrm{T}_0 K(Y) \to \mathbb{Z}$ (compare to [7, Corollary 6.10 of Chapter II]). Note that the composition $\mathrm{T}_0 K(Y) \to \mathrm{T}_0 K(\bar{Y}) \to \mathbb{Z}$ of the restriction homomorphism with the degree homomorphism for $\bar{Y}$ coincides with the degree homomorphism for Y.

Since $\bar{X}$ is isomorphic to a direct product of projective spaces, the homomorphism $\deg\colon \mathrm{T}_0\, K(\bar{X}) \to \mathbb{Z}$ is bijective. In particular, since $\mathrm{T}_0\, K(X)$ is a non-zero subgroup of $\mathrm{T}_0\, K(\bar{X})$, we see that the quotient $\mathrm{T}_0\, K(\bar{X})/\mathrm{T}_0\, K(X)$ is finite.

(1) If E is a common splitting field of the algebras $A_1, \dots, A_n$, the variety X_E has a closed rational point. Therefore, there exists a zero-cycle on X of degree $[E : F]$. It follows that the order of the quotient $\mathrm{T}_0\, K(\bar{X})/\mathrm{T}_0\, K(X)$ divides $[E : F]$. In particular, $v_p([E : F]) \geq \beta'$.

(2) It follows from the definition of β' and the above discussion that there exists a zero-cycle $\sigma = \sum_{i=1}^{r} l_i\sigma_i$ on X (where $l_i \in \mathbb{Z}$ and $\sigma_i \in X$) with $v_p(\deg(\sigma)) = \beta'$. Since

$$\deg(\sigma) \overset{\text{def}}{=} \sum_{i=1}^{r} l_i \deg(\sigma_i),$$

one has $v_p(\deg(\sigma_i)) \leq \beta'$ for certain i. Denote by E the residue field of the point σ_i. Since the variety X_E possess a rational point, E is a common splitting field of the algebras $A_1, \dots, A_n$. Therefore, by item (1), $v_p([E : F]) \geq \beta'$. On the other hand, $v_p([E : F]) = v_p(\deg(\sigma_i)) \leq \beta'$. Thus $v_p([E : F]) = \beta'$. ∎

Proof of Theorem III.5: (1) Let $x_1, \dots, x_n$ be some elements with the behaviour α in the Brauer group of a field F. Consider the variety X as in Definition III.3. According to item 2 of Lemma III.6, the elements $x_1, \dots, x_n$ possess a common splitting field E/F with $v_p([E : F]) = \beta' \overset{\text{def}}{=} v_p(|\,\mathrm{T}_0\, K(\bar{X})/\mathrm{T}_0\, K(X)|)$. On the other hand, $\beta = v_p(\Gamma_0 K(\bar{X})/\Gamma_0 K(X))$ by Lemma III.4. Since

$$\mathrm{T}_0\, K(\bar{X}) = \Gamma_0 K(\bar{X}) \quad \text{and} \quad \mathrm{T}_0\, K(X) \supset \Gamma_0 K(X)$$

(see [6, Theorem 3.9 of Chapter V] for the second relation) the order of the quotient $\mathrm{T}_0\, K(\bar{X})/\mathrm{T}_0\, K(X)$ divides the order of the quotient $\Gamma_0 K(\bar{X})/\Gamma_0 K(X)$. Therefore $\beta' \leq \beta$ and consequently $v_p([E : F]) \leq \beta$.

(2) Let $x_1, \dots, x_n$ be the Brauer classes of some "generic" division algebras with the behaviour α (which exist by Proposition III.1). Let X be the product of the Severi–Brauer varieties of these division algebras. By item (1) of Lemma III.6, $v_p([E : F]) \geq \beta'$ for any common splitting field E/F of $x_1, \dots, x_n$. By [11, Théorème 5.5], one has $\mathrm{T}_0\, K(X) = \Gamma_0 K(X)$. Therefore $\beta' = \beta$. ∎

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